Department of Mathematics

Pattamundai College, Pattamundai

4^{th} Semester

Ring Theory and Linear Algebra

Core – 10

Sec–A

(Unit–1)

1. {0} and R are subrings of any ring R. {0} is called the ____________ sub ring of R.

2. {0, 2, 4} is a sub ring of the ring Z_6, the integers module 6.

3. Find all zeros of x^2+3x+2 in Z_6.

4. What is the characteristic of the following ring

   i) R = 2Z (ii) R = $Z_3 \times Z_4$

5. Let a belong to a ring R. Let S = $\{x \in R \mid ax = 0\}$. Show that S is a sub ring of R.

6. Find the characteristic of the ring R = $Z_2 \times Z_3 \times Z_4$.

7. Write the elements of the factor ring $\frac{Z}{4Z}$.

8. Find all maximal ideals in Z_{12}.

9. Show the following statement is true / false. {0} is a prime ideal in Z_6.

10. How many elements are in $\frac{Z[i]}{< 3+i >}$?

11. Give an example of a Boolean ring of infinite order.

12. How many idempotent element in z_{256}?

(Unit–2)

13. How many ring homomorphism from Z to Z.

14. How many ring homomorphism from Z_m to Z_m.

15. How many ring homomorphism from z_8 to z_4.

16. Show that the correspondence f(x) = 5x from z_5 to z_{10}. does not preserve addition.

17. Is the ring 2Z isomorphic to the ring 3Z.

18. Determine all ring homomorphism from Q to Q.

19. Write only the statement of 1st isomorphism theorem of the ring.

20. Write only the statement of 2nd isomorphhism theorem of the ring.

P.T.O.

21. Write only the statement of 3rd isomorphism theorem of the ring.

22. Write the definition of field of Quotients.

23. If R is a ring with unity and the characteristic of R is n>0, then R contains a sub ring isomorphic to z_n.

24. If R is a ring with unity and the characterisitc of R is zero then R contains a subring isomorphic to z. (True/False)

25. Show that kerϕ is a sub ring of the ring homomorphism $\phi : R \rightarrow R^1$

26. Find all the solution of the congruence $2x \cong 6$ (mod4).

27. Find all the solution of the congruence $3x \cong 1$ (mod 5).

28. The sequence 2, 10, 18, 26..... contains no cube. (T/F)

(Unit–3)

29. Write only the definition of vector space.

30. The union of two vector space is a vector space? Justify your answer.

31. Let $A = \{(x_1, x_2, x_3\} \in R^3 \mid x_1x_2 = 0|$ is a vector space of R^3?

32. Show that A = {(1, 2, 3)}, (4,5,0), (6, 0, 0)} is a linear independent or linear dependent set.

33. If a set is linear independent then any subset of it is also linearly independent set. Justify your answer.

34. The basis of a vector space is unique? (T/F)

35. Find the dimension of the matrix $(A)_{m \times n}$.

36. Find the dimension of the symmetric matrix of order 'n'.

37. Find the dimension of the anti-symmetric matrix of order 'n'.

38. If U and W are two sub spaces of a finite dimensional vector space v, then

    dim $(U+W)$= dim U + dim w – dim $(U \cap W)$. (T/F)

39. Find a basis for a subspace U of V in

    $U = \{(x_1,x_2,x_3,x_4,x_5) \in V_5 | \ x_1+x_2+x_3= 0, 3x1 - x_4+7x_5 =0 \}$, and $V = V_5$.

40. If U and W are subspace ofa finite dimensional vector space V such that $U \cap W = \{0\}$ then dim $(U \oplus W)$ = dimU+ dim w. (T/F).

(Unit–4)

41. There exist a linear transformation $T : V_2 \rightarrow V_4$ such that T(0,0) = (1, 00.0)?

42. If T (x+y) = T(x) + T(y) then T is linear. (T/F)

43. If T : $R^2 \to R^2$ is defined by $T(a_1,a_2) = (1, a_2)$ is not a linear transformation.

44. If T : $R^2 \to R^2$ is linear, T(1,0) = (1,4) and T (1,1) = (2,5) What is T (2,3) = ?

45. Let V and W be finite dimensional vector spaces and T : V→W be mean, if dim(v) < dim w then T is onto?

46. Let v be a vector space over IF. Then V is isomorphic to IF^n if and only if dim(v) =n. (T/F).

47. Let V and W be finite dimensional vector spaces. Then V is isomorphic to w iff dim((v) = dim w (T/F)

48. Let T : V→W be a linear map $([T]_\alpha^\beta)^{-1} = [T^{-1}]_\alpha^\beta$ (T/F).

49. Prove that a linear transformation on a one-dimensional vector space is nothing but multiplication by a fixed scalar.

50. Let T:U→ V be a map such that T (Ou) ≠ Ov. Then T is not linear. Justify your answer.

Sec–B

(Unit–1)

1. Show that the set $M_2(z)$ of 2×2 matrices with integral entries is a non commutative ring with unity $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2. Let a, b belong to a ring R. Then show that (–a) (–b) = ab

3. The set of Gaussian integers Z[i] = {a+ib, a,b∈z} is a subring of the complex number $\mathbb{C}$. Explain it.

4. Show that Z×Z is not an integral domain.

5. Let a, b, and c belong to an integral domain. If a≠0 and ab = ac then b=c.

6. Determine all units in the ring Z×Z.

7. Show that the characteristic of an integral domain is zero or prime.

8. Show that the ideal <x> a prime ideal in Z[x] but not a maximal ideal in z[x].

9. Show that A = {(3x,y)| x, y∈z} is a maximal ideal of Z⊕Z.

10. Let R be a ring with unity. If I is an ideal of R containing a unity of R then show that I =R.

11. Let S = {a+bi/ a, b∈z, b is even}. Then show that S is a sub ringh of Z [i] but not an ideal of z[i].

12. show that $\dfrac{z_3[x]}{<x^2+x+1>}$ is not a field.

(Unit–2)

13. Let ϕ : Z→Z defined by ϕ (x) = 0.x. Prove that ϕ is a ring homomorphism.

14. Let ϕ : Z→Z, defined by ϕ (x) = 2x. Prove that ϕ is not a ring homomorphism.

15. Let ϕ is a ring homomorphism, $\phi : Z_7 \to Z_7$. Then compute $\phi[(x^4+2x)(x^3-3x^2+3)]$.

P.T.O.

16. Consider the mapping from $M_2(Z)$ into Z given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to a$. Prove or disprove that this is a ring homomorphism.

17. Is the mapping from Z_{10} to Z_{10} give by $x \to 2x$ a ring homomorphism?

18. Let n be an integer with decimal representation ak $a_{k-1} \ldots a_1 a_0$. Prove that n is divisible by 3 if and only if $ak + a_{k-1} + \ldots + a_1 + ao$ is divisible by 3.

19. Let 'n' be an integer with decimal representation ak $a_{k-1} \ldots a_1 a_0$. Prove that 'n' is divisible by 4 if and only if $a_1 a_0$ is divisible by 4.

20. Let ϕ be a ring homomorphism. For any $r \in R$ and any positive integer n, $\phi(nr) = n\phi(r)$ and $\phi(r^n) = (\phi(r))^n$.

21. Let ϕ be a ring homomorphism. Then prove that If A is an ideal and ϕ is onto R^1, then $\phi(A)$ is an ideal.

22. If ϕ is an isomorphism from R to R^1, then ϕ^{-1} is an isomorphism from R^1 to R.

23. Let : ϕ $R \to R^1$ be a homomorphism then prove that ker $\phi = \{r \in R | \phi(r) = 0\}$ is an ideal of R.

24. Show that a ring homomorphism $\phi : R \to R^1$ is one-one mapping if and only if $\ker\phi = \{e\}$.

25. Show that the form f: $Z_5 \to Z_{10}$ given by $f(x) = 3x$ is not a homomorphism.

26. How many ring homomorpism from z to z×z

27. Show that Q is not isomorphic to IR.

28. Show that Q[i] not isomorphic to Q.

(Unit–3)

29. In any vector space V

a) $\alpha.0 = 0$ for every scalar α

b) $0.4 = 0$ for every $u \in V$.

30. Let L be the set of all vectors of the form (x,2x, –3x, x) in V_4. Then prove that L is a subspae of V_4.

31. Show that the set S of all polynomials $P \in$ **p**, which vanishes at a fixed point x_0, is a sub space of **p**.

32. Show that if S is a nonempty subset of a vector space V, then [S] is the smallest sub space of V containing S.

33. If S is a nonempty subset of a vector space V, prove that

i) [S] = S if and only if S is a subspace of V.

ii) [[S]] = [S].

34. Let V be any vector space. Then prove that

a) The set {v} is L.D iff v= 0

b) The set { v_1, v_2 } is L.D iff v_1 and v_2 are collinear.

35. If v has a basis of 'n' elements, then every other basis for V also has 'n' elements.

36. Let V =V_3, U = the xy-plane and w= the yz plane in V_3. Then find the dim $(U \cap W)$

37. Show that the intersection of two vector space is a vector space

38. Let U be a sub space of a finite dimensional vector space V. Then dim U $\leq$ dim V. Equality holds only when U=V.

39. In a vector space V let B = {v_1, $v_2 v_3$......... v_n} span V. Then the following two conditions are equivalent

a) {v_1, $v_2 v_3$......... v_n} is a linearly independent set.

b) If $\mu \in V$, then the expression $v = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n$ is unique.

(Unit–4)

40. Let $T : R^2 \rightarrow R^2$ by $T(a_1, a_2) = (2a_1 + a_2, a_1)$. To show that T is linear.

41. Let $T : M_{m\times n}^{(IF)} \rightarrow M_{n\times m}^{(IF)}$. defined by $T(A) = A^T$, where A^T is te transpose of A. Show that T is Linear.

42. Let V and W be vector spaces and T : V$\rightarrow$W be linear. Then N(T) and R(T) are sub spaves of V and W, respectively.

43. Let the linear transformation T is defined by T: $P_2(R) \rightarrow M_{2\times 2}(R)$ $T(f(x)) = \begin{pmatrix} f(l) - f(2) & 0 \\ 0 & f(0) \end{pmatrix}$. Then show that dim (R(T)) = 2.

44. Let V and W be vector spaces and Let T :V$\rightarrow$W be linear. Then prove that T is one-one if and only if N(T)= {0}.

45. Let $T : P_2(R) \rightarrow P_2(R)$ be a linear transformation defined by $T(f(x)) = 2 f^1(x) + 3\int_0^x f(-)dt$. Then find R(T) and N(T).

46. Let T : U$\rightarrow$V be a non singular linear map. Then prove that T^{-1}: V$\rightarrow$U is a linear, one-one and onto map.

47. If T : U$\rightarrow$V is a linear map, where U is finite dimensional, prove that

a) n(T) $\leq$ dim U.

b) r(T) $\leq$ min (dimU, dimV)

48. If U and V are finite dimensional vector spaces of same dimension, then a linear map T:U$\rightarrow$V is one-one if and only if T is onto.

P.T.O.

49. Let $T:R^2 \to R^3$ be the linear transformation defined by $T(a_1,a_2) = (a_1+3a_2, 0, 2a_1 - 4a_2)$ Let β and Υ be the standard order basis for R^2 and R^3. respectively. Then find the a matrix $[T]_\beta \Upsilon$.

50. Let $D : P_3 \to P_3$ be the differential map $D(P) = P^1$. Let us calculate the marix of D relative to the standard bases $\{1, x, x^2, x^3\}$

Sec – C

(Unit–1)

1. Prove that a finite integral domain in a field.

2. Show that the Boolean ring is commutative ring.

3. Let $Z_3[i] = \{a+ib | a,b \in z_3\}$. Show that $z_3[i]$ is a field with 9 elements.

4. Prove that the only ideals of a field are {0} and F itself.

5. Consider the factor ring of the Gaussian integers. $R = \dfrac{z[i]}{<2-i>}$ Show that R is a field and $R \cong z_5$.

(Unit–2)

6. Let $\phi : Z_2 \to Z_2$ defined by $\phi(x) = x^2 - x$ is a ring homomorphism.

7. State and prove first isomorphism theorem for rings.

8. Let D be an integral domain. Then there exists a field IF that contains a subring isomorphic to D.

9. Let $Z[\sqrt{2}] = \{a+b\sqrt{2} / a,b \in Z\}$ Let $H = \left\{ \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \mid a,b \in z \right\}$ show that $z[\sqrt{2}]$ and H are isomorphic as rings.

10. Let $R = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix}, a,b \in Z \right\}$ and let ϕ be the mapping that takes $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ to a–b.

a) Show that ϕ is a homomorphism.

b) Determine the Kernel of ϕ.

c) Show that $\dfrac{R}{Ker\phi} \cong Z$.

d) Is $Ker(\phi)$ a prime ideal?

e) Is $ker(\phi)$ a maximal ideal?

(Unit–3)

11. Let R^+ be the set of all positive real numbers. Define the operations of addition and scalar multiplication as follows:

i) $u+v = u.v$ for all $u,v \in R^+$

ii) $\alpha.u = u^\alpha$ for all $u \in R^+$ and Scalar α. Prove that R^+ is a real vector space.

12. Let U and W be two sub spaces of a vector space V and Z =U+W. Then prove that $Z = U \oplus W$ if and only if the following condition is eatisfied. Any vector $z \in Z$ can be expressed uniquely as the sum $z=u+w$, $u \in U$, $w \in W$.

13. In a vector space V if $\{v_1, v_2....v_n\}$ generates V and if $\{w_1, w_2, w_3, ...w_m\}$ is L.I then $m \leq n$.

14. If U and W are two subspaces of a finite dimensional vector space V, then prove that

$\dim (U+W) = \dim (U) + \dim (W) - \dim (U \cap W)$.

15. Let = V = P_3. the space of all real polynomials of degree at most 3. Let U be the sub space of P_3 consisting of those polynomials of P_3 that vanish at x =1, Let w be the sub space of P_3 consisting of those polynomials of P3 whose first derivatives vanish at x =1. Then find basis and dimension of the sub space U, W, $U \cap W$, and U+W.

(Unit – 4)

16. Let V and W be vector spaces and Let $T:V \to W$ be linear. If V is finite-dimensional, then prove that N(T)+R(T) = dimV or nultity(T) + rank(T) = dim V.

17. Let V and W be vector spaces over the field F and suppose that $\{ v_1, v_2, v_3, v_n \}$ is a basis for V. For $w_1, w_2, w_3 ...... w_n$, in W, there that (vi) = wi for i= 1, 2, 3 ...n. Justify your answer.

18. Let V be a finite dimensional vector space and let $T:V \to V$ be linear. Then prove that

a) If rank(T) = rank (T^2) then R(T) n N(T) = {0}

b) for some positive integer K, prove that $V = R(T^K) \oplus N(T^K)$.

19. Prove that the linear map $T : V_3 \to V_3$ defined by $T(e_1) = e_1 - e_2$ $T(e_2) = 2e_2 + e_3$., $T(e_3) = e_1 + e_2 + e_3$ is neither one-one nor on to.

20. Let $T : P_4 \to P_4$ be a linear transformation defined by $T(P)(x) = \int_1^x P'(t)\, dt$. Then determine the matrix $[T]_\beta^y$

$\beta = \Upsilon = \{1, x, x^2, x^3, x^4\}$

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